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ASYMPTOTIC STABILITY OF SPR_SODE MODEL FOR DENGUE

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ABSTRACT

An ordinary differential equation with stochastic parameters, called SPR_SODE model for the spread of dengue fever was considered to analyze further. It was defined the set of stochastic equations and a reproductive number R_0. This R_0 was defined for mosquito as well as human parameters. In this paper, the asymptotic stability of the disease-free equilibrium point of the above said model was discussed.

KEYWORDS: SPR_SODE Model, Stochastic, Scaled Variables, Asymptotic Stability, Reproductive Number

INTRODUCTION

Dengue is one of the diseases which is in worldwide and comes under infectious diseases. The work of a carrier (i.e) the medium for transmitting is performed by the mosquito, "Aedes Ageypti [4, Gantmacher.F.R., 1977]. There are so many models for such infectious diseases. We need a separate model for such a special disease like dengue fever for better results. In this work, the SPR_SODE model [2,3, Dhevarajan.S, et.al, 2013] (SPR_Stochastic Ordinary differential model) is considered to analyze further. Asymptotically stable equilibrium points or equilibrium solutions can be defined as the equilibrium solutions in which solutions that start "near" them move toward the equilibrium solution [1, Boyd et.al, 1999].

SPR_SODE MODEL

All the notions of SPR_SODE model [2,3, Dhevarajan.S, et.al, 2013] are taken for further analysis without any change and the same model is given below.

$$\frac{d}{dt}\left[SS_h\right]=\wp_h+b_h\left[\left[TP\right]_h\right]+L_h\left[\left[RC\right]_h\right]-\tau_h(t)\left[SS\right]_h-\Omega_h\left[\left[TP\right]_h\right]\left[SS\right]_h$$

$$\frac{d}{dt}\left[EX_h\right]=\tau_h(t)\left[SS\right]_h-\xi_h\left[EX\right]_h-\Omega_h\left(\left[TP\right]_h\right)\left[EX\right]_h$$

$$\frac{d}{dt}\left[IF_h\right]=\xi_h\left[EX\right]_h-\left[RC\right]_h\left[IF\right]_h-\Omega_h\left(\left[TP\right]_h\right)\left[RC\right]_h$$

$$\frac{d}{dt}\left[RC_h\right]=\theta_h\left[IF\right]_h-L_h\left[RC\right]_h-\Omega_h\left(\left[TP\right]_h\right)\left[IF\right]_h-\eta\left[IF\right]_h$$

$$\frac{d}{dt}\left[SS_m\right]=[BIR]_m\left[TP\right]_m-\tau_m(t)\left[SS\right]_m-\Omega_m\left(\left[TP\right]_m\right)\left[SS\right]_m$$

$$\frac{d}{dt}\left[EX_m\right]=\tau_m(t)\left[SS\right]_m-\xi_m\left[EX\right]_m-\Omega_m\left(\left[TP\right]_m\right)\left[EX\right]_m$$

$$\frac{d}{dt}\left[IF_m\right]=\xi_m\left[EX\right]_m-\Omega_m\left(\left[TP\right]_m\right)\left[IF\right]_m \qquad \text{(A)}$$

By converting (A) to fractional quantities and denoting each scaled population by small letters, one can get,

$$\frac{d}{dt}\left[ex\right]_h = \frac{\phi_h \phi_m P_{mh}[if]_m}{\phi_m\left[TP\right]_m + \phi_h\left[TP\right]_h}.\left[TP\right]_m.\left[1-[ex]_h-[if]_h-[re]_h\right]-\left[\xi_h+[BIR]_h+\frac{\wp_h}{\left[TP\right]_h}\right][ex]_h+\eta_h[if]_h[ex]_h$$

$$\frac{d}{dt}\left[if\right]_h = \xi_h\left[ex\right]_h - \left(\theta_h + \left[BIR\right]_h + \frac{\wp_h}{\left[TP\right]_h}\right)\left[if\right]_h + \eta_h\left[if\right]_h^2$$

$$\frac{d}{dt}\left[rc\right]_h = \theta_h\left[if\right]_h - \left(L_h + [BIR]_h\frac{\wp_h}{\left[TP\right]_h}\right)\left[rc\right]_h + \eta_h\left[if\right]_h\left[TP\right]_h$$

$$\frac{d}{dt}\left[TP\right]_h = \wp_h + \theta_h\left[TP\right]_h - \left([DID]_h + [DDD]_h\left[TP\right]_h\right).\left[TP\right]_h - \eta_h[if]_h[TP]_h$$

$$\frac{d}{dt}\left[ex\right]_m = \frac{\phi_h\phi_m}{\phi_m\left[TP\right]_m + \phi_h\left[TP\right]_h}.\left[TP\right]_h.\left[P_{mh}[if]_h + P_{mh}[rc]_h\right].\left[1-[ex]_h-[if]_h\right]-\left[\xi_h+[BIR]_m\right][ex]_m$$

$$\frac{d}{dt}\left[if\right]_m = \xi_m\left[ex\right]_m - [BIR]_m\left[if\right]_m$$

$$\frac{d}{dt}\left[TP\right]_m = [BIR]_m\left[TP\right]_m - \left([DID]_m + [DDD]_m\left[TP\right]_m\right)\left[TP\right]_m \tag{B}$$

Now, R_0 can be defined as, $R_0 = \sqrt{\Re_{hm}.\Re_{mh}}$, (C)

where $\Re_{hm}$ and $\Re_{mh}$ can be written in mathematical notation as,

$$\Re_{hm} = \frac{\xi_m}{\xi_m + [DID]_m + [DDD]_m\left[TP\right]_m^*}\frac{\phi_h\phi_m P_{mh}\left[TP\right]_h^*}{\phi_m\left[TP\right]_m^* + \phi_h\left[TP\right]_h^*}P_{mh}\left[[DID]_m + [DDD]_m\left[TP\right]_m^*\right]^{-1}$$

$$\begin{aligned}\Re_{mh} = &\frac{\xi_h}{\xi_h + [DID]_h + [DDD]_h\left[TP\right]_h^*}\frac{\phi_h\phi_m P_{mh}\left[TP\right]_m^*}{\phi_h\left[TP\right]_h^* + \phi_m\left[TP\right]_m^*}\left(\theta_h + \eta_h + [DID]_h + [DDD]_h\left[TP\right]_h^*\right)^{-1}\\ &.\left[P_{mh} + \overline{P_{mh}}.\theta_h\left(\eta_h + [DID]_m + [DDD]_m\left[TP\right]_h^*\right)^{-1}\right]\end{aligned} \tag{D}$$

ASYMPTOTIC STABILITY

The Jacobian of the dengue model (B) evaluated at x_{nodis} is of the form

$$J = \begin{pmatrix} J_{11} & 0 & 0 & 0 & 0 & J_{16} & 0 \\ J_{21} & J_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & J_{12} & J_{13} & 0 & 0 & 0 & 0 \\ 0 & J_{42} & 0 & J_{44} & 0 & 0 & 0 \\ 0 & J_{52} & 0 & J_{54} & J_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{65} & J_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J_{77} \end{pmatrix} \tag{1}$$

Now our interest to find out the eigen values of (1). Let the eigen values be λ_i, i=1, 2, 3,..,7. The fourth and seventh columns of the jacobian given by (1) corresponding to the total human and mosquito populations, contain only the diagonal terms. The diagonal terms of the Jacobian (1) provide two of the eigen values say, λ_1 and λ_2 and can be defined by,

$$\lambda_1 = [BIR]_h - [DID]_h - 2[DDD]_h [TP]_h^* = -\sqrt{\left[[BIR]_h - [DID]_h\right]^2 + 4[DDD]_h \wp_h} \tag{2}$$

$$\lambda_2 = [BIR]_m - [DID]_m - 2[DDD]_m [TP]_m^* = -\left[[BIR]_m - [DID]_m\right] \tag{3}$$

Let us conclude from the earlier assumption that, both λ_1 and λ_2 are always negative since $[BIR]_m > [DID]_m$. Let us create another matrix by excluding the fourth and seventh rows and columns of the Jacobian (1). By solving the characteristic equation of the matrix formed now, gives the other five eigen values. Now by defining U_i's as $U_1 = \xi_h + \left[BIR\right]_h + \frac{\wp_h}{[TP]_h^*}$, $U_2 = \theta_h + \eta_h + \left[BIR\right]_h + \frac{\wp_h}{[TP]_h^*}$, $U_3 = L_h + \eta_h + \left[BIR\right]_h + \frac{\wp_h}{[TP]_h^*}$, $U_4 = \xi_m + \left[BIR\right]_m$, $U_5 = \left[BIR\right]_m$, $U_6 = [MB]_h^* \,.\, P_{mh}$, $U_7 = \xi_h$, $U_8 = [MB]_m^* \,.\, P_{hm}$, $U_9 = \xi_m$, $U_{10} = \theta_h$, $U_{11} = [MB]_m^* \,.\, \overline{P_{hm}}$, Where, W_i's can be written as,

$$W_5 = 1\,;\; W_4 = U_1 + U_2 + U_3 + U_4 + U_5\,;\; W_4 = U_1 + U_2 + U_3 + U_4 + U_5$$

$$W_3 = U_1U_2 + U_1U_3 + U_1U_4 + U_1U_5 + U_2U_3 + U_2U_4 + U_2U_5 + U_3U_4 + U_3U_5 + U_4U_5,$$

$$W_2 = U_1U_2U_3 + U_1U_2U_4 + U_1U_2U_5 + U_1U_3U_4 + U_1U_3U_5 + U_1U_4U_5 + U_2U_3U_4 + U_2U_3U_5 + U_2U_4U_5 + U_3U_4U_5,$$

$$W_1 = U_1U_2U_3U_4 + U_1U_2U_3U_5 + U_1U_2U_4U_5 + U_1U_3U_4U_5 + U_2U_3U_4U_5 - U_6U_7U_8U_9,$$

$$W_0 = U_1U_2U_3U_4U_5 - U_3U_6U_7U_8U_9 - U_6U_7U_9U_{10}U_{11}$$ With

$$W_5\lambda^5 + W_4\lambda^4 + W_3\lambda^3 + W_2\lambda^2 + W_1\lambda + A_0 = 0 \tag{4}$$

Now, we have to find out the signs of the solutions of (4). The Liénard – Chipart criterion [6] gives, any of the following four conditions is necessary and sufficient in order that all roots of a polynomial $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0$ with real coefficients have negative real parts:

$a_n > 0,\ a_{n-2} > 0, ...\Delta_1 > 0, \Delta_3 > 0, ...$; $a_n > 0,\ a_{n-2} > 0, ...\Delta_2 > 0, \Delta_4 > 0, ...$;

$a_n > 0,\ a_{n-1} > 0, a_{n-3} > 0, ...\Delta_1 > 0, \Delta_3 > 0, ...$; $a_n > 0,\ a_{n-1} > 0, a_{n-3} > 0, ...\Delta_2 > 0, \Delta_1 > 0, ...$, where Δ_i be the principal minors with order i, with i = 1, 2, 3,…,n. For the use the Routh–Hurwitz criterion, First it is to prove that when $R_0 < 1$, all roots of (4) have negative real part. The Routh–Hurwitz criterion [5, section 1.6-6(b)] states that for a real algebraic equation

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0 \tag{5}$$

Given $a_n > 0$, all roots have negative real part if and only if $a_n = V_0$, $a_{n-1} = V_1$, $V_2 = \begin{vmatrix} a_{n-1} & a_n \\ a_{n-3} & a_{n-2} \end{vmatrix}$, $V_3 = \begin{vmatrix} a_{n-1} & a_n & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-5} & a_{n-4} & a_{n-3} \end{vmatrix}$, $\ldots V_n = \begin{vmatrix} a_{n-1} & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & a_0 \end{vmatrix}$ are all positive, with $a_i = 0$ for i < 0. This is true if and only if all a_i and either all even-numbered V_k or all odd-numbered V_k are positive (6, Li´enard–Chipart test). By Korn and Korn [5] in section 1.6-6(c) state Descartes's rule of sign as the number of positive real roots of a real algebraic equation (5) is equal to the number; N_a, of sign changes in the sequence, $a_n, a_{n-1}, \ldots, a_1, a_0$ of coefficients, where the departure terms are ignored, or it is less than N_a by a positive even integer.

Now, it is to show that when $R_0 < 1$, all the coefficients, W_i, of the characteristic equation (4), and V_0, V_2, and V_4, are positive, hence by the Routh–Hurwitz criterion one can say that, all the eigen values of (1) have negative real part. Now, it is to show that when $R_0 > 1$, there is one and only one sign change in the sequence $W_5, W_4 \ldots W_0$ hence, by Descartes's rule of sign along with positive real part, there is only one eigen value, and also the disease free equilibrium point is unstable. The expression for R_0^2 in (C) can be written, in terms of Ui, as

$$R_0^2 = \frac{U_3U_6U_7U_8U_9 + U_6U_7U_9U_{10}U_{11}}{U_1U_2U_3U_4U_5} \tag{6}$$

For $R_0 < 1$, by (6), $U_3U_6U_7U_8U_9 + U_6U_7U_9U_{10}U_{11} < U_1U_2U_3U_4U_5$ (7)

$$U_6U_7U_8U_9 < U_1U_2U_4U_5 \tag{8}$$

As all the $B_{i'}$'s are positive, W_5, W_4, W_3 and W_2 are always positive. From (8) we see that $W_1 > 0$, and from (7) we see that $W_0 > 0$. Thus, for $R_0 < 1$, all Wi are positive. We now show that the even-numbered V_k are positive for $R_0 < 1$. For the fifth-degree polynomial (4), $V_0 = W_5$, which is always positive. $V_2 = W_3W_4 - W_2W_5$, which we can show to be a positive sum of products of U_i's, so $V_2 > 0$. Lastly, $V_4 = W_1[W_2W_3W_4 - (W_1W_4^2 + W_2^2\ W_5)] - W_0[W_3(W_3W_4 - W_2W_5) - (2W_1W_4W_5 - W_0W_2^5)]$. For convenience let us use the notations T_1 and T_2 such that $T_1 = [W_2W_3W_4 - (W_1W_4^2 + W_2^2\ W_5)]$ and $T_2 = [W_3(W_3W_4 - W_2W_5) - (2W_1W_4W_5 - W_0W_2^5)]$ respectively. Where $T_1 > 0$ and $T_2 > 0$. Hence $V_4 = W_1T_1 - W_0T_2$. Let us define $T_2^{[1]} = T_2 + U_6U_7U_9U_{10}U_{11}$. As $T_2^{[i]} > T_2$ and $W_0 > 0$ for $V_4^{[1]} = W_1T_1 - W_0T_2^{[1]}, V_4 > V_4^{[1]}$. Similarly, let us define $W_0^{[1]} = W_0 - U_3U_6U_7U_8U_9 - U_6U_7U_9U_{10}U_{11}$. As $W_4 > W_4^{[1]}$ and $T_2^{[1]} > 0$, for $V_4^{[2]} = W_1T_1 - W_0^{[1]}T_2^{[1]}$, $V_4^{[1]} > V_4^{[2]}$ at last, let us define $W_1^{[1]} = W_1 - U_1U_2U_3U_5 + U_6U_7U_8U_9$, As $W_1^{[1]} < W_1$ (for $R_0 < 1$) and $T_1 > 0$ For $V_4^{[3]} = W_1^{[1]}T_1 - W_0^{[1]}T_2^{[1]}$, $V_4^{[2]} > V_4^{[3]}$. It can be shown that $V_4^{[3]}$ Is a sum of positive terms, so $V_4^{[3]} > 0$ $V_4^{[1]} > V_4^{[2]} > V_4^{[3]}$, $V_4 > 0$. Thus, for $R_0 < 1$, all roots of (4) have negative real parts. When $R_0 > 1$, $U_3U_6U_7U_8U_9 + U_6U_7U_9U_{10}U_{11} < U_1U_2U_3U_4U_5$ and so W0 < 0. As W_5, W_4, W_3 and W_2 are positive, the sequence, W_5, W_4, W_3, W_2, W_1 has exactly one sign change. Thus, by Descartes's rule of sign, (4) has one positive real root when R_0 > 1.

CONCLUSIONS

A stochastic ordinary differential equation called SPR_SODE model for the spread of dengue fever is analyzed. For our model, the disease-free equilibrium point, x_{nodis}, is locally asymptotically stable if R_0 < 1 and unstable if R_0 > 1. If R_0 < 1, on average each infected individual infects less than one other individual, and the disease dies out. If R_0 > 1, on average each infected individual, infects more than one other individual, so one can expect the disease to spread. The Jacobian of (B) at x_{nodis} has one eigen value equal to 0 at R_0 = 1.

REFERENCES

1. Boyd, John P. "The Devil's Invention: Asymptotic, Super asymptotic and Hyper asymptotic Series". Acta Applicandae Mathematicae 56 (1): 1–98, March 1999.
2. S.Dhevarajan, A.Iyemperumal, S.P.Rajagopalan and D.Kalpana, "SPR_SODE Model for dengue fever", International Journal of applied mathematical and statistical sciences", ISSN 2319-3972. Vol. 2, Issue 3, 41-46, July 2013.
3. S.Dhevarajan, A.Iyemperumal, S.P.Rajagopalan and D.Kalpana, "Improved SPR_SODE Model for dengue fever", International Journal of Advanced Scientific and Technical Research", ISSN 2249-9954. Vol. 5, Issue 3, 418-425, Oct. 2013.
4. F.R.Gantmacher, "The theory of matrices", 1, Chelsea, reprint (1977) (Translated from Russian)
5. G.A.Korn, and T.M.Korn, "Mathematical Handbook for Scientists and Engineers: Definitions, Theorems, and Formulas for Reference and Review", Dover Publications, Mineola, NY, 2000.
6. A.Liénard and H.Chipart, "Sur la signe de la partie réelle des racines d'une équation algébrique" J. Math. Pures Appl., 10, 1914, pp. 291–346.
7. P.Reiter, "Texas Lifestyle Limits Transmission of Dengue Virus", Emerging Infectious Diseases, 9, 2003, pp. 86-89.